

# Foundations of Programming

Probabilities and Naïve Bayes models

# Announcements

- Depth & Breadth
  - Depth: Data Mutation, nested loops, if-statements and pictures
  - Breadth: ??
- Diversity in Computing lunch form:  
<https://goo.gl/forms/OqIkiuhmOQbWTQWs2>

# Learning outcomes/key ideas

- Basics of probability
- Representing Joint Probability
- Inference by enumeration
- Bayes Rule
- Naïve Bayes for sentiment classification

# Text sentiment classification

 93%

 47%

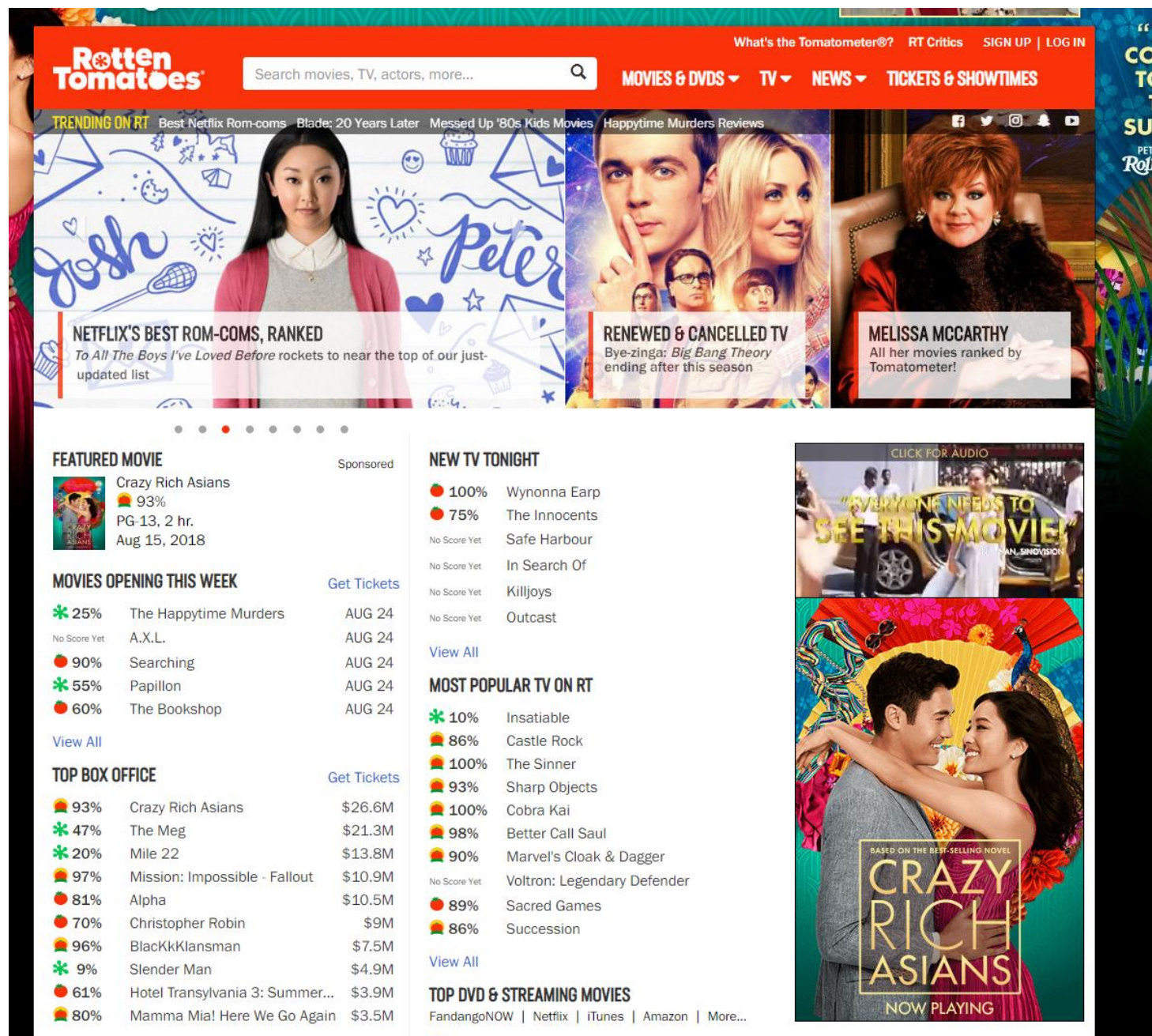
 20%

 97%

 81%

 70%

 96%



The screenshot shows the Rotten Tomatoes website interface. At the top, there's a navigation bar with the Rotten Tomatoes logo, a search bar, and links for MOVIES & DVDS, TV, NEWS, and TICKETS & SHOWTIMES. Below the navigation bar, there are several featured sections:

- TRENDING ON RT:** Includes "Best Netflix Rom-coms", "Blade: 20 Years Later", "Messed Up '80s Kids Movies", and "Happytime Murders Reviews".
- NETFLIX'S BEST ROM-COMS, RANKED:** Features a list of rom-coms with their scores and release dates.
- RENEWED & CANCELLED TV:** Lists TV shows that have been renewed or cancelled, such as "Big Bang Theory".
- MELISSA MCCARTHY:** A section dedicated to her movies, ranked by Tomatometer.
- FEATURED MOVIE:** "Crazy Rich Asians" is featured with a score of 93% and a release date of August 15, 2018.
- MOVIES OPENING THIS WEEK:** Lists movies opening on August 24th, such as "The Happytime Murders" (25%), "A.X.L." (No Score Yet), "Searching" (90%), "Papillon" (55%), and "The Bookshop" (60%).
- TOP BOX OFFICE:** Lists the top-grossing movies, including "Crazy Rich Asians" (\$26.6M), "The Meg" (\$21.3M), "Mile 22" (\$13.8M), "Mission: Impossible - Fallout" (\$10.9M), "Alpha" (\$10.5M), "Christopher Robin" (\$9M), "BlacKkKlansman" (\$7.5M), "Slender Man" (\$4.9M), "Hotel Transylvania 3: Summer..." (\$3.9M), and "Mamma Mia! Here We Go Again" (\$3.5M).
- NEW TV TONIGHT:** Lists TV shows airing tonight, such as "Wynonna Earp" (100%), "The Innocents" (75%), "Safe Harbour" (No Score Yet), "In Search Of" (No Score Yet), "Killjoys" (No Score Yet), and "Outcast" (No Score Yet).
- MOST POPULAR TV ON RT:** Lists popular TV shows, including "Insatiable" (10%), "Castle Rock" (86%), "The Sinner" (100%), "Sharp Objects" (93%), "Cobra Kai" (100%), "Better Call Saul" (98%), "Marvel's Cloak & Dagger" (90%), "Voltron: Legendary Defender" (No Score Yet), "Sacred Games" (89%), and "Succession" (86%).
- TOP DVD & STREAMING MOVIES:** Lists movies available on streaming services like FandangoNOW, Netflix, iTunes, and Amazon.

On the right side of the page, there are several promotional banners for movies and TV shows, including "EVERYONE NEEDS TO SEE THIS MOVIE" and "CRAZY RICH ASIANS".

# Basic Probability Theory

- An **experiment** has a set of potential outcomes, e.g., throw a dice
- The **sample space** of an experiment is the set of all possible outcomes, e.g.,  $\{1, 2, 3, 4, 5, 6\}$ 
  - A **random variable** can take on any value in the sample space
- An **event** is a subset of the sample space.
  - $\{2\}$
  - $\{3, 6\}$
  - $\text{even} = \{2, 4, 6\}$
  - $\text{odd} = \{1, 3, 5\}$

# Probability as Relative Frequency



Total Flips: 10

Number Heads: 5

Number Tails: 5

Probability of Heads:

Number Heads / Total Flips = 0.5

Probability of Tails:

Number Tails / Total Flips = 0.5 = 1.0 – Probability of Heads

The experiments, the sample space  
and the events must be defined  
clearly for probability to be  
meaningful

# Probabilities and classification

We are trying to determine that the probability that a given movie review is positive vs. negative. We will select the classification that has a higher probability.

One quantity we care about in this task is the **prior probability** of the sentiment of a review. In other words:

$P(\textit{Sentiment} = \textit{pos})$  and  $P(\textit{Sentiment} = \textit{neg})$

True or false:  $P(\textit{Sentiment} = \textit{pos}) + P(\textit{Sentiment} = \textit{neg}) = 1$

A. True   B. False   C. It depends

# Prior probability

Of 100 students completing a course, 20 were business major. Ten students received As in the course, and three of these were business majors.

What is the probability that a randomly selected student got an A?

$P(A = \text{True}) = ?$

- A. 0.03   B. 0.1   C. 0.3   D. 0.2   E. None of these



# Joint probability: Two or more events BOTH happening

Of 100 students completing a course, 20 were business major. Ten students received As in the course, and three of these were business majors.

What is the probability that a randomly selected student is a Business student and gets an A?

$P(A = \text{True}, B = \text{True}) = ?$

A. 0.03   B. 0.1   C. 0.3   D. 0.2   E. None of these

*"the probability of A and B"*

	B(usiness student) = True	B(usiness student) = False
A = True		
A = False		

$P(A=\text{True}) = P(A=\text{True}, B=\text{True}) + P(A=\text{True}, B=\text{False})$

*marginalization*

Will the values in a joint probability table always sum to 1?

A. Yes   B. No

# But how does evidence change things?

## *Conditional* probability

Of 100 students completing a course, 20 were business major. Ten students received As in the course, and three of these were business majors. What is the probability of A after knowing B is true?

What is the probability that a randomly selected *business student* gets an A?

$P(A = \text{True} \mid B = \text{True}) = ?$

A. 0.03   B. 0.1   C. 0.3   D. 0.2   E. None of these

*"the probability of A given B"*

	B(usiness student) = True	B(usiness student) = False
A = True	0.03	0.07
A = False	0.17	0.73

$$P(A=\text{True} \mid B = \text{True}) = P(A=\text{True}, B=\text{True}) / P(B=\text{True})$$

# Inference

- Often we will be able to measure some information, but we want to make statements about things that are not directly in our table of information.
- E.g., we want to know what is the probability of a toothache in general (the prior probability), never mind whether there's a cavity, catch, etc.

	<i>toothache</i>		$\neg$ <i>toothache</i>	
	<i>catch</i>	$\neg$ <i>catch</i>	<i>catch</i>	$\neg$ <i>catch</i>
<i>cavity</i>	<b>.108</b>	<b>.012</b>	<b>.072</b>	<b>.008</b>
$\neg$ <i>cavity</i>	<b>.016</b>	<b>.064</b>	<b>.144</b>	<b>.576</b>

Inference = Using known facts to derive others

# Inference by enumeration

- Start with the joint probability distribution:

	<i>toothache</i>		$\neg$ <i>toothache</i>	
	<i>catch</i>	$\neg$ <i>catch</i>	<i>catch</i>	$\neg$ <i>catch</i>
<i>cavity</i>	<b>.108</b>	<b>.012</b>	<b>.072</b>	<b>.008</b>
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- For any proposition  $\phi$ , sum the atomic events where it is true:  $P(\phi) = \sum_{\omega: \omega \models \phi} P(\omega)$

# Inference by enumeration

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<i>cavity</i>	<b>.108</b>	<b>.012</b>	.072	.008
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- For any proposition  $\phi$ , sum the atomic events where it is true:  $P(\phi) = \sum_{\omega: \omega \models \phi} P(\omega)$
- $P(\textit{toothache}) = 0.108 + 0.012 + 0.016 + 0.064 = 0.2$

# Inference by enumeration

- Start with the joint probability distribution:

	<i>toothache</i>		$\neg$ <i>toothache</i>	
	<i>catch</i>	$\neg$ <i>catch</i>	<i>catch</i>	$\neg$ <i>catch</i>
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- Can also compute conditional probabilities:

$$\begin{aligned} P(\neg \text{cavity} \mid \text{toothache}) &= \frac{P(\neg \text{cavity} \wedge \text{toothache})}{P(\text{toothache})} \\ &= \frac{0.016 + 0.064}{0.108 + 0.012 + 0.016 + 0.064} \\ &= 0.4 \end{aligned}$$

# Normalization

	<i>toothache</i>		$\neg$ <i>toothache</i>	
	<i>catch</i>	$\neg$ <i>catch</i>	<i>catch</i>	$\neg$ <i>catch</i>
<i>cavity</i>	<b>.108</b>	<b>.012</b>	<b>.072</b>	<b>.008</b>
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These terms are a pain to compute

# Normalization

	<i>toothache</i>		$\neg$ <i>toothache</i>	
	<i>catch</i>	$\neg$ <i>catch</i>	<i>catch</i>	$\neg$ <i>catch</i>
<i>cavity</i>	<b>.108</b>	<b>.012</b>	<b>.072</b>	<b>.008</b>
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$$P(\textit{cavity} \mid \textit{toothache}) = \frac{0.12}{0.108 + 0.012 + 0.016 + 0.064}$$

$$P(\neg \textit{cavity} \mid \textit{toothache}) = \frac{0.08}{0.108 + 0.012 + 0.016 + 0.064}$$



# Normalization

	<i>toothache</i>		$\neg$ <i>toothache</i>	
	<i>catch</i>	$\neg$ <i>catch</i>	<i>catch</i>	$\neg$ <i>catch</i>
<i>cavity</i>	<b>.108</b>	<b>.012</b>	<b>.072</b>	<b>.008</b>
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$$P(\text{cavity} \mid \text{toothache}) = \alpha 0.12$$

$$P(\neg \text{cavity} \mid \text{toothache}) = \alpha 0.08$$

$$\alpha = \frac{1}{0.108 + 0.012 + 0.016 + 0.064}$$

$$P(\neg \text{cavity} \mid \text{toothache}) + P(\text{cavity} \mid \text{toothache}) = \alpha 0.08 + \alpha 0.12 = ??$$

What value is ??

- A. 1
- B.  $\alpha$
- C.  $0.108 + 0.012 + 0.016 + 0.064$
- D. Something else
- E. I have no idea!

# Normalization

	<i>toothache</i>		$\neg$ <i>toothache</i>	
	<i>catch</i>	$\neg$ <i>catch</i>	<i>catch</i>	$\neg$ <i>catch</i>
<i>cavity</i>	<b>.108</b>	<b>.012</b>	<b>.072</b>	<b>.008</b>
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$$P(\text{cavity} \mid \text{toothache}) + P(\neg \text{cavity} \mid \text{toothache}) = \alpha 0.12 + \alpha 0.08 = 1$$

$$\alpha = 1 / (0.12 + 0.08)$$

$$\alpha = 1 / 0.2 = 5$$

$$\langle P(\text{cavity} \mid \text{toothache}), P(\neg \text{cavity} \mid \text{toothache}) \rangle = \langle \alpha 0.12, \alpha 0.08 \rangle = \langle 0.6, 0.4 \rangle$$

$$\text{Compare to computing: } \alpha = \frac{1}{0.108 + 0.012 + 0.016 + 0.064}$$

# Bayes Theorem

If  $P(E_2) > 0$ , then

$$P(E_1 | E_2) = P(E_2 | E_1)P(E_1) / P(E_2)$$

This can be derived from the definition of conditional probability.

# Bayes Rule example

A patient takes a lab test and the result comes back positive. The test has a false negative rate of 2% and false positive rate of 3%. Furthermore, 0.8% of the entire population have this cancer.

What is the probability of cancer if we know the test result is positive?

"False negative" = Considering only situations where there is cancer, test is negative "False positive" = Considering only situations where there is not cancer, test is positive
--

# Bayes Rule example

A patient takes a lab test and the result comes back positive. The test has a false negative rate of 2% and false positive rate of 3%. Furthermore, 0.8% of the entire population have this cancer.

What is the probability of cancer if we know the test result is positive?

We know:

We want:

"False negative" = Considering only situations where there is cancer, test is negative  
"False positive" = Considering only situations where there is not cancer, test is positive

# Bayes Rule example

A patient takes a lab test and the result comes back positive. The test has a false negative rate of 2% and false positive rate of 3%. Furthermore, 0.8% of the entire population have this cancer.

What is the probability of cancer if we know the test result is positive?

We know:

$$P(\text{Test=pos} | \neg\text{cancer}) = 0.03 \text{ FP}$$

$$P(\text{Test=neg} | \text{cancer}) = 0.02 \text{ FN}$$

$$P(\text{cancer}) = 0.008 \text{ Prior}$$

We want:

$$P(\text{cancer} | \text{Test=pos})$$

# Bayes Rule example

A patient takes a lab test and the result comes back positive. The test has a false negative rate of 2% and false positive rate of 3%. Furthermore, 0.8% of the entire population have this cancer.

What is the probability of cancer if we know the test result is positive?

We know:

$$P(\text{Test=pos} | \neg\text{cancer}) = 0.03 \text{ FP}$$

$$P(\text{Test=neg} | \neg\text{cancer}) = 0.97$$

$$P(\text{Test=pos} | \text{cancer}) = 0.98$$

$$P(\text{Test=neg} | \text{cancer}) = 0.02 \text{ FN}$$

$$P(\text{cancer}) = 0.008 \text{ Prior}$$

$$P(\neg\text{cancer}) = 0.992$$

We want:

$$P(\text{cancer} | \text{Test=pos}) =$$

$$\frac{P(\text{Test=pos} | \text{cancer})P(\text{cancer})}{P(\text{Test=pos})} = \mathbf{0.21}$$

$$P(\text{Test=pos}) = P(\text{Test=pos, cancer}) + P(\text{Test=pos, } \neg\text{cancer})$$

$$P(\text{Test=pos, cancer}) = P(\text{Test=pos} | \text{cancer})P(\text{cancer})$$

$$P(\text{Test=pos, } \neg\text{cancer}) = P(\text{Test=pos} | \neg\text{cancer})P(\neg\text{cancer})$$

# Returning to review sentiment classification...

Our evidence is the text in the review. We want to estimate

$$P(\textit{Sentiment} = \textit{pos} \mid \textit{text})$$

But we could never directly estimate this because we're unlikely to have ever seen this specific text before! How can Bayes rule help us?

$$P(\textit{Sentiment} = \textit{pos} \mid \textit{text}) = \frac{P(\textit{text} \mid \textit{Sentiment} = \textit{pos}) P(\textit{Sentiment} = \textit{pos})}{P(\textit{text})}$$



# Returning to review sentiment classification...

$$P(\textit{Sentiment} = \textit{pos} \mid \textit{text}) \propto P(\textit{text} \mid \textit{Sentiment} = \textit{pos}) P(\textit{Sentiment} = \textit{pos})$$

Simplifying assumption 1: Represent text with a "bag of words" representation

$$P(w_1 = \textit{true}, w_2 = \textit{true}, w_3 = \textit{false}, \dots \mid \textit{Sentiment} = \textit{pos})$$

But can we learn this??

# Independence: Intuition

- Events are independent if one has nothing whatever to do with others. Therefore, for two independent events, knowing one happening does not change the probability of the other event happening.
  - one toss of coin is independent of another coin (assuming it is a regular coin).
  - price of tea in England is independent of the result of general election in Canada.

# Independence: Definition

- Events A and B are independent iff:

$$P(A, B) = P(A) \times P(B)$$

which is equivalent to

$$P(A | B) = P(A) \text{ and}$$

$$P(B | A) = P(B)$$

when  $P(A, B) > 0$ .

T1: the first toss is a head.

T2: the second toss is a tail.

$$P(T2 | T1) = P(T2)$$

# Conditional Independence

- Dependent events can become independent given certain other events.
- Example,
  - Size of shoe
  - Size of vocabulary
  - ??
- Two events  $A$ ,  $B$  are conditionally independent given a third event  $C$  iff
$$P(A|B, C) = P(A|C)$$

# Conditional Independence: Utility via Naïve Bayes

- Let  $E_1$  and  $E_2$  be two events, they are conditionally independent given  $E$  iff  $P(E_1 | E, E_2) = P(E_1 | E)$ ,  
that is the probability of  $E_1$  is not changed after knowing  $E_2$ , given  $E$  is true.
- Equivalent formulations:  
 $P(E_1, E_2 | E) = P(E_1 | E) P(E_2 | E)$   
 $P(E_2 | E, E_1) = P(E_2 | E)$

$$P(w_1 = \text{true}, w_2 = \text{true}, w_3 = \text{false}, \dots | \text{Sent} = \text{pos}) \\ = P(w_1 = \text{true} | \text{Sent} = \text{pos}) P(w_2 = \text{true} | \text{Sent} = \text{pos}) P(w_3 = \text{false} | \text{Sent} = \text{pos}) \dots$$

A ha! These we can learn from data!

# Returning to review sentiment classification...

$$P(\textit{Sentiment} = \textit{pos} \mid \textit{text}) \propto P(\textit{text} \mid \textit{Sentiment} = \textit{pos}) P(\textit{Sentiment} = \textit{pos})$$

Simplifying assumption 1: Represent text with a "bag of words" representation

$$P(w_1 = \textit{true}, w_2 = \textit{true}, w_3 = \textit{false}, \dots \mid \textit{Sentiment} = \textit{pos})$$

Simplifying assumption 2: Words are conditionally independent given sentiment

$$\begin{aligned} & P(w_1 = \textit{true}, w_2 = \textit{true}, w_3 = \textit{false}, \dots \mid \textit{Sent} = \textit{pos}) \\ = & P(w_1 = \textit{true} \mid \textit{Sent} = \textit{pos}) P(w_2 = \textit{true} \mid \textit{Sent} = \textit{pos}) P(w_3 = \textit{false} \mid \textit{Sent} = \textit{pos}) \dots \end{aligned}$$

# Training a Naïve Bayes classifier

$$P(\textit{Sentiment} = \textit{pos} \mid \textit{text}) \propto P(\textit{text} \mid \textit{Sentiment} = \textit{pos}) P(\textit{Sentiment} = \textit{pos})$$

Simplifying assumption 1: Represent text with a "bag of words" representation

$$P(w_1 = \textit{true}, w_2 = \textit{true}, w_3 = \textit{false}, \dots \mid \textit{Sentiment} = \textit{pos})$$

Simplifying assumption 2: Words are conditionally independent given sentiment

$$\begin{aligned} & P(w_1 = \textit{true}, w_2 = \textit{true}, w_3 = \textit{false}, \dots \mid \textit{Sent} = \textit{pos}) \\ = & P(w_1 = \textit{true} \mid \textit{Sent} = \textit{pos}) P(w_2 = \textit{true} \mid \textit{Sent} = \textit{pos}) P(w_3 = \textit{false} \mid \textit{Sent} = \textit{pos}) \dots \end{aligned}$$

# Practice with Probability

- Which of the following statements are generally true? (If they are true only in certain conditions, state what the conditions are)

$$P(A,B) = P(A)*P(B)$$

$$P(A,B) = P(A|B)$$

$$P(A,B) = P(A|B)P(B)$$

$$P(A|B) + P(A|\neg B) = 1$$

$$P(\neg A) + P(A) = 1$$

$$P(\neg A, B) + P(A, B) = P(B)$$

$$P(\neg A|B) + P(A|B) = P(B)$$